SOME OTHER ALGEBRAIC PROPERTIES OF FOLDED HYPERCUBES

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Abstract. We construct explicity the automorphism group of the folded hypercube FQ_n of dimension $n > 3$, as a semidirect product of N by M, where N is isomorphic to the Abelian group Z_2^n , and M is isomorphic to $Sym(n+1)$, the symmetric group of degree $n+1$, then we will show that the folded hypercube FQ_n is a symmetric graph.

Keywords : Hypercube; 4-cycle; Linear extension ; Permutation group; Semidirect product; Symmetric graph

AMS Subject Classifications: 05C25; 94C15

1. Introduction and Preliminaries

A folded hypercube is an edge transitive graph, this fact is the main result that has been shown in [8]. In this note, we construct explicity the automorphism group of a folded hypercube, then we will show that a folded hypercube is not only an edge transitive graph, but also a symmetric graph. In this paper, a graph $G = (V, E)$ is considered as an undirected graph where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set. For all the terminology and notation not defined here, we follow [2, 3, 5]. The hypercube Q_n of dimension n is the graph with vertex-set $\{(x_1, x_2, ..., x_n)|x_i \in \{0, 1\}\}$, two vertices $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$ are adjacent if and only if $x_i = y_i$ for all but one i. The folded hypercube FQ_n of dimension n, proposed first in [1], is a graph obtained from the hypercube Q_n by adding an edge, called a complementary edge, between any two vertices $x = (x_1, x_2, ..., x_n)$, $y = (\bar{x_1}, \bar{x_2}, ..., \bar{x_n})$, where $\bar{1} = 0$ and $\bar{0} = 1$. The graphs shown in Fig. 1, are the folded hypercubes FQ_3 and FQ_4. The graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are called isomorphic,

if there is a bijection $\alpha : V_1 \longrightarrow V_2$ such that, $\{a, b\} \in E_1$ if and only if $\{\alpha(a), \alpha(b)\} \in E_2$ for all $a, b \in V_1$. In such a case the bijection α is called an isomorphism. An automorphism of a graph Γ is an isomorphism of Γ with itself. The set of automorphisms of Γ, with the operation of composition of functions, is a group, called the automorphism group of Γ and denoted by $Aut(\Gamma)$. A permutation of a set is a bijection of it with itself. The group of all permutations of a set V is denoted by $Sym(V)$, or just $Sym(n)$ when $\mid V \mid = n$. A permutation group G on V is a subgroup of $Sym(V)$. In this case we say that G acts on V. If Γ is a graph with vertex-set V, then we can view each automorphism as a permutation of V, so $Aut(\Gamma)$ is a permutation group. Let G acts on V, we say that G is transitive (or G acts transitively on V) if there is just one orbit. This means that given any two elements u and v of V, there is an element β of G such that $\beta(u) = v$.

The graph Γ is called vertex transitive if $Aut(\Gamma)$ acts transitively on $V(\Gamma)$. For $v \in V(\Gamma)$ and $G = Aut(\Gamma)$, the stabilizer subgroup G_v is the subgroup of G containing all automorphisms which fix v. In the vertex transitive case all stabilizer subgroups G_v are conjugate in G, and consequently isomorphic, in this case, the index of G_v in G is given by the equation, $|G : G_v| = \frac{|G|}{|G_v|} = |V(\Gamma)|$. If each stabilizer G_v is the identity group, then every element of G, except the identity, does not fix any vertex, and we say that G acts semiregularly on V. We say that G acts regularly on V if and only if G acts transitively and semiregularly on V and in this case we have $\mid V \mid = \mid G \mid$. The action of $Aut(\Gamma)$ on $V(\Gamma)$ induces an action on $E(\Gamma)$ by the rule $\beta\{x, y\} = \{\beta(x),\ \beta(y)\}, \beta \in Aut(\Gamma)$, and Γ is called edge transitive if this action is transitive.The graph Γ is called symmetric, if for all vertices $u, v, x, y,$ of Γ such that u and v are adjacent, and x and y are adjacent, there is an automorphism α such that $\alpha(u) = x, and, \alpha(v) = y$. It is clear that a symmetric graph is vertex transitive and edge transitive.

Let G be any abstract finite group with identity 1, and suppose that Ω is a set of generators of G, with the properties :

(i) $x \in \Omega \Longrightarrow x^{-1} \in \Omega; (ii) 1 \notin \Omega$;

The Cayley graph $\Gamma = \Gamma(G, \Omega)$ is the graph whose vertex-set and edge-set defined as follows : $V(\Gamma) = G; E(\Gamma) = \{\{g, h\} \mid g^{-1}h \in \Omega\}$.

It can be shown that the hypercube Q_n is the Cayley graph $\Gamma(Z_2^n, B)$, where $B = \{e_1, e_2, ..., e_n\}$, e_i is the element of Z_2^n with 1 in the $i-th$ position and 0 in the other positions for, $1 \leq i \leq n$. Also, the folded hypercube FQ_n is the Cayley graph $\Gamma(Z_2^n, S)$, where $S = B \cup \{u = e_1 + e_2 + ... + e_n\}$. Hence the hypercube Q_n and the folded hypercube FQ_n are vertex transitive graphs. Since Q_n is Hamiltonian [6] and a spanning subgraph of FQ_n, so FQ_n is Hamiltonian. Some properties of the folded hypercube FQ_n are discussed in $[6, 7, 8]$.

The group G is called a semidirect product of N by Q, denoted by $G = N \rtimes Q$, if G contains subgroups N and Q such that, (i)$N \trianglelefteq G$ (N is a normal subgroup of G); (ii) $NQ = G$; (iii) $N \cap Q = \{1\}$.

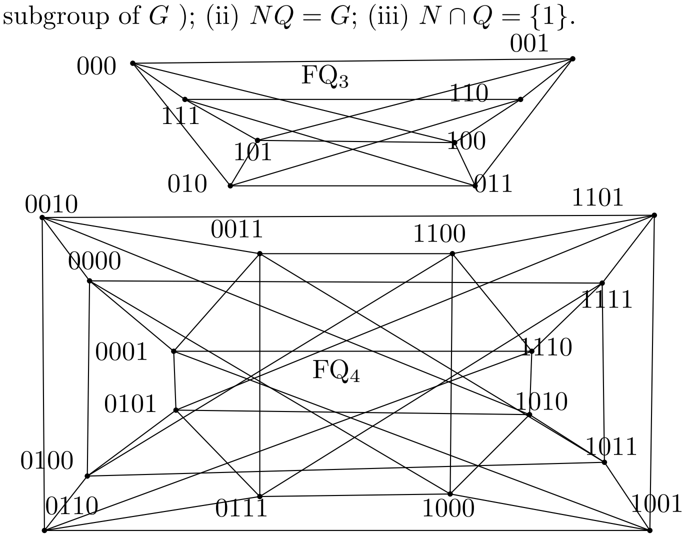

Fig. 1. The folded hypercubes FQ_3 and FQ_4.

2. **Main results**

Lemma 2.1. *If $n \neq 3$, then every 2-path in FQ_n is contained in a unique 4-cycle.*

Proof. If $n = 2$, then it is trivial that the assertion of the Lemma is true, so let $n > 3$. Let $P : uvw$ be a 2-path in FQ_n. If $u = (x_1, ..., \bar{x_i}, ..., x_n)$, $v = (x_1, ..., x_i, ..., x_n)$, $w = (x_1, ..., \bar{x_j}, ..., x_n)$, then only vertex $x = (x_1, ..., x_{i-1}, \bar{x_i}, ..., x_{j-1}, \bar{x_j}, ..., x_n)$ and v are adjacent to both vertices u and w. Hence the 4-cycle $C : uvwx$ is the unique 4-cycle that contains the 2-path P. If $u =$

$(\bar{x_1}, \bar{x_2}, ..., \bar{x_n})$, $v = (x_1, ..., x_i, ..., x_n)$, $w = (x_1, ..., x_{j-1}, \bar{x_j}, x_{j+1}, ..., x_n)$, then only vertices $x = (\bar{x_1}, ..., \bar{x_{j-1}}, x_j, \bar{x_{j+1}}, ..., \bar{x_n})$ and v are adjacent to both vertices u and w.

□

In the folded hypercube FQ_3 any 2-path is contained in 3 4-cycles, hence the assertion of Lemma 2.1 is not true in FQ_3.

Remark 2.2. For a graph Γ and $v \in V(\Gamma)$, let $N(v)$ be the set of vertices w of Γ such that w is adjacent to v. Let $G = Aut(\Gamma)$, then G_v acts on $N(v)$, if we restrict the domains of the permutations $g \in G_v$ to $N(v)$. Let L_v be the set of all elements g of G_v such that g fixes each element of $N(v)$. Let $Y = N(v)$ and $\Phi : G_v \longrightarrow Sym(Y)$ be defined by the rule, $\Phi(g) = g_{|Y}$ for any element g in G_v , where $g_{|Y}$ is the restriction of g to Y. In fact Φ is a group homomorphism and $ker(\Phi) = L_v$, thus G_v/L_v and the subgroup $\phi(G_v)$ of $Sym(Y)$ are isomorphic. If $\mid Y \mid= deg(v) = k$, then $\mid G_v \mid / \mid L_v \mid\leq k!$.

Lemma 2.3. *If $n > 3$ and $G = Aut(FQ_n)$, then $|G| \leq (n + 1)!2^n$*

Proof. Let $v \in V(FQ_n)$ and L_v be the subgroup which is defined in the above, we show that $L_v = \{1\}$. Let $g \in L_v$ and w be an arbitrary vertex of FQ_n. If the distance of w from v is 1, then w is in $N(v)$, so $g(w) = w$. Let the distance of w from v be 2. Then there is a vertex u such that $P : vuw$ is a 2-path, hence by Lemma 2.1. there is a 4-cycle that contains this 2-path, thus there is a vertex t such that $C : tvuw$ is a 4-cycle. Since $t \in L_v$, then $g(t) = t$, so $g(C) : tvug(w)$ is a 4-cycle. By Lemma 2.1 the 2-path $P_1 : tvu$ is contained in a unique 4-cycle, thus $g(C) = C$, therefore $g(w) = w$. The set S is a generating set for the Abelian group Z_2^n, so the Cayley graph $FQ_n = \Gamma(Z_2^n, S)$ is a connected graph. Now, by induction on the distance w from v, it follows that $g(w) = w$, so $g = 1$ and $L_v = \{1\}$. Now, by the Remark 2.2. , $|G_v| \leq |L_v|(n + 1)! \leq (n + 1)!$.

The folded hypercube FQ_n is a vertex transitive graph, hence $|G| = |G_v||V(FQ_n)| \leq (n + 1)!2^n$.

□

Theorem 2.4. *If $n > 3$, then $Aut(FQ_n)$ is a semidirect product of N by M, where N is isomorphic to the Abelian group Z_2^n and M is isomorphic to the group $Sym(n+1)$.*

Proof. Let $Aut(FQ_n) = G$, $v \in Z_2^n = V(FQ_n)$ and ρ_v be the mapping $\rho_v : Z_2^n \longrightarrow Z_2^n$ defined by $\rho_v(x) = v + x$. Since FQ_n is the Cayley graph $\Gamma(Z_2^n, S)$, then ρ_v is an automorphism of FQ_n and $N = \{\rho_v | v \in Z_2^n\}$ is a subgroup of G isomorphic to Z_2^n. Note that the Abelian group Z_2^n is also a vector space over the field $F = \{0, 1\}$ and $B = \{e_1, e_2, ..., e_n\}$ is a basis of this vector space and any subset of the set $S = B \cup \{u = e_1 + e_2 + ... + e_n\}$ with n elements is linearly independent over F and is a basis of the vector space Z_2^n. Let A be a subset of S with n elements and $f : B \longrightarrow A$ be a one to one function. We can extend f over Z_2^n linearly. Let ϕ be the linear extension of f over Z_2^n, thus ϕ is a linear mapping of the vector space Z_2^n into itself such that $\phi_{|B} = f$. Since B and A are bases of the vector space Z_2^n, hence ϕ is a permutation of Z_2^n. In fact ϕ is an automorphism of FQ_n. If $A = B$, then $\phi(u) = \phi(e_1) + \phi(e_2) + ... + \phi(e_n) = e_1 + e_2 + ... + e_n = u$. If $A \neq B$, then $u \in A$ and for some $i, j \in \{1, 2, ..., n\}$ we have $\phi(e_i) = u$ and $e_j \notin A$. Then $\phi(u) = \phi(e_1) + \phi(e_2) + ... + \phi(e_n) = e_1 + e_2 + ... + e_{j-1} + e_{j+1} + ... + e_n + u = u - e_j + u = e_j \in S$. Now, it follows that ϕ maps S into S. If $[v, w] \in E(FQ_n)$, then $w = v + s$ for some $s \in S$, hence $\phi(w) = \phi(v) + \phi(s)$, now since $\phi(s) \in S$ we have $[\phi(v), \phi(w)] \in E(FQ_n)$. For a fixed n-subset A of S there are $n!$ distinct one to one functions such as f, thus there are $n!$ automorphisms of the folded hypercube FQ_n such as ϕ. The set S has $n+1$ elements, so there are $n+1$ n-subset of S such as A, hence there are $(n+1)!$ one to one functions $f : B \longrightarrow S$. Let $M = \{\phi : Z_2^n \longrightarrow Z_2^n \mid \phi$ is a linear extension of a one to one function $f : B \longrightarrow S\}$. Then M has $(n+1)!$ elements and any element of M is an automorphism of FQ_n. If $\alpha \in M$, then α maps S onto S, hence $\alpha_{|S}$, the restriction of α to S, is a permutation of S. Now it is an easy task to show that M is isomorphic to the group $Sym(S)$. Every element of M fixes the element 0, thus $N \cap M = \{1\}$, hence $|MN| = \frac{|M||N|}{|N \cap M|} = (2^n)(n+1)!$, therefore $|Aut(FQ_n)| \geq (2^n)(n+1)!$. Now, by the Lemma 2.3. it follows that $|Aut(FQ_n)| = (n+1)!2^n$, therefore $Aut(FQ_n) = MN$.

We show that the subgroup N is a normal subgroup of $Aut(FQ_n) = G = MN = NM$. It is enough to show that for any $f \in M$ and $g \in N$, we have $f^{-1}gf \in N$. There is an element $y \in Z_2^n$ such that $g = \rho_y$. Let b be an arbitrary vertex of FQ_n, then $f^{-1}gf(b) = f^{-1}\rho_y f(b) = f^{-1}(y+f(b)) = f^{-1}(y) + b = \rho_{f^{-1}(y)}(b)$, hence $f^{-1}gf = \rho_{f^{-1}(y)} \in N$.

□

It is an easy task to show that the folded hypercube FQ_3 is isomorphic to $K_{4,4}$, the complete bipartite graph of order 8, so $Aut(FQ_4)$ is a group with $2(4!)^2 = 1152$ elements [2], therefore Theorem 2.3 is not true for $n = 3$.

If $n > 1$, then the assertion of Lemma 2.1 is also true for the hypercube Q_n and by a similar method that has been seen in the proof of Theorem 2.4. we can show that $Aut(Q_n) \cong Z_2^n \rtimes Sym(n)$, the result which has been discussed in [4] by a different method.

Theorem 2.5. *If $n \geq 2$, then the folded hypercube FQ_n is a symmetric graph.*

Proof. The folded hypercube FQ_2 is isomorphic to K_4, the complete graph of order 4, and the folded hypercube FQ_3 is isomorphic to $K_{4,4}$, the complete bipartite graph of order 8, both of these are clearly symmetric. Let $n \geq 4$. Since The folded hypercube FQ_n is a Cayley graph, then it is vertex transitive, now it is sufficient to show that for a fixed vertex v of $V(FQ_n)$, G_v acts transitively on $N(v)$, where $G = Aut(FQ_n)$. As we can see in the proof of Theorem 2.3, since each element of M is a linear mapping of the vector space ${Z_2}^n$ over $F = \{0, 1\}$, then for the vertex $v = 0$ the stabilizer group of G_v is M. The restriction of each element of M to $N(0) = S$ is a permutation of S. If $f \in M$ fixes each element of S, then f is the identity mapping of the vector space ${Z_2}^n$. Since $|S| = n + 1$, then $Sym(S)$ has $(n + 1)!$ elements. On the other hand $\bar{M} = \{f_{|S} \; | f \in M\}$ has $(n + 1)!$ elements, hence $\bar{M} = Sym(S) = G_0$. We know that $Sym(X)$ acts transitively on X, where X is a set, so G_0 acts transitively on $N(0)$.

□

Corollary 2.6. *The connectivity of the folded hypercube FQ_n is maximum, say $n + 1$.*

Proof. Since the folded hypercube FQ_n is a symmetric graph, then it is edge transitive, on the other hand this graph is a regular graph of valency $n+1$. We know that the connectivity of a connected edge transitive graph is equal to its minimum valency [3, pp. 55].

□

The above fact has been rephrased in [1] and has been found in a different manner.

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